Distribution theory December 4, 2002.

Exam paper[1]

I

Consider the differential operator $D = (\frac{d}{dx})^2 + \frac{d}{dx} - 2$.

1. Determine the fundamental solution of D belonging to $\mathcal{D}'_+ = \{T \in \mathcal{D}'(\mathbb{R}) : \mathrm{supp}(T) \subset [0, +\infty)\}$.

2. Determine the solution $T \in \mathcal{D}'_+$ of the equation $DT = Y$ (Y the Heaviside one-step function). Hint: use convolution or operational calculus.

3. Determine, with the help of the Heaviside operational calculus, the solution f of the following classical initial value problem, where g is a given continuous function defined on $\mathbb{R}$:

$$Df = g, \; f(0) = 0, \; f'(0) = 1. \tag{1}$$

4. Find the solution f in the case where $g = 1$.

5. Let G be a function of the class C^2 on $\mathbb{R}$. Give conditions on G implying that there exists a continuous function f on $\mathbb{R}$ which satisfies the following convolution equation (2). Determine in that case the solution f.

$$\int_0^x f(x-y)(e^y - e^{-2y})dy = G(x), \qquad x \geq 0 \tag{2}$$

II

1. Define the notion of convergence of distributions $T_n \to T$, $n \in \mathbb{N}$ or $T = \lim_{n\to\infty} T_n$, where $T_n, T \in \mathcal{D}'(\mathbb{R})$.

2. Show that the map $\frac{d}{dx} : T \mapsto T'$ is continuous from $\mathcal{D}'(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R})$, that is, $T_n \to T$ implies $T'_n \to T'$.

3. Give an example of a sequence of continuously differentiable functions $(f_n)_{n\geq 1}$ such that $T_{f_n} \to \delta$.

4. Give a sequence of continuous functions $(g_n)_{n\geq 1}$ such that $T_{g_n} \to \delta'$.

III

1. Give the definition of 'tempered distribution', and of the Fourier transform of a tempered distribution.

2. For $T \in \mathcal{S}'(\mathbb{R})$ give the Fourier transforms of T' and of $2\pi i x T$.

3. Show that the Fourier transform of an even (respectively odd) distribution is even (respectively odd).

4. Determine the Fourier transform T of the distribution $S = \mathrm{sign}$ (the function equal to 1 for $x > 0$, and to -1 for $x < 0$).

5. Give the general inversion formula for the Fourier transform of tempered distributions. By means of this determine the Fourier transform of the distribution $\mathrm{pv}\frac{1}{x}$.

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[1] The parts I, II and III are independent. Clarify your answers by stating the theorems used.